## Very rudimentary foundations of mathematics

### for advanced high-school students

This brief essay was the consequence of a discussion I had with the Head of Mathematics at Sherborne School in 1981. I had suggested that the School Maths Project might consider including a short discussion piece about the foundations of mathematics. Nothing other than this essay came out of that discussion. The initial paragraphs, dealing with matrices and Cayley, were intended as a bridge between the pupils' earlier algebra studies and the subsequent material on foundations. I emphasise that the essay is a *very* rudimentary introduction to the foundations of mathematics.

Most of the material we have been studying in A-level algebra was first discovered last century. Curiously, determinants were first studied before matrices; it was only around the middle of the nineteenth century that A. Cayley, J.J. Sylvester and others began to look at the array of numbers (the matrix) in a determinant as an object to be examined on its own. The word 'matrix' was first used in this context by Sylvester, in 1850, although Cayley, whose first paper on matrices appeared in 1855, is generally considered to be the founder of matrix theory.

As a schoolboy, Arthur Cayley (1821-95) showed sufficient promise to persuade his teachers and his father to send him to Cambridge instead of the family business. After a distinguished mathematical career as a student at Cambridge, Cayley was elected a Fellow of Trinity College, a position he vacated after only three years in order to avoid taking Holy Orders. He then spent fifteen years as a professional lawyer, at the same time publishing nearly two hundred original papers in mathematics. Finally, he was appointed to a chair of mathematics at Cambridge, and was able to devote his full working time to teaching and research in many areas of the subject. He is particularly famous for his contributions to matrix theory, projective geometry, and invariant theory, and was one of the most prolific mathematicians in history: only Cauchy and Euler have published more. In spite of this, Cayley had a wide range of leisure interests, including literature, painting, and architecture. To Cayley is also due the first presentation of the axioms of abstract group theory (1849),[1] which you studied in the A-level course.

Let's look more closely at the axiomatic method. The basic idea of the method is to write down certain statements—the *axioms*—which we accept as true in the theory we are developing, and from which we deduce theorems systematically. Before we write down our axioms, we must agree on two matters: one, the rules of argument—the logic—that we shall use; the other, the objects that we shall accept as requiring no definition (the undefined terms).

[1] Lest you imagine that such objects as matrices and groups are of no importance outside the context of pure mathematics, here is a remark of the eminent nineteenth century physicist P.G. Tait, a remark made about matrix theory but nowadays equally applicable to group theory: 'Cayley is forging the weapons for future generations of physicists.'

Surprisingly, not all mathematicians agree on the rules of argument which they will allow in proofs. Indeed, since L.E.J. Brouwer (1881-1966) published his thesis [3] in 1907, there has been considerable controversy over this point. However, it is fair to say that most mathematicians today work with the logic known as *classical logic*.

To illustrate the idea of undefined terms, we turn to geometry, where we have such undefined terms as *point*, *line* and so on. In fact, when Euclid wrote his famous *Elements of Geometry* around 300BC, he attempted to define some of these terms. For example, he defined a point to be 'that which has no part', and a line to be 'a breadthless length'. Perhaps you will agree that it is better to leave point and line as undefined terms, rather than make such cumbersome, unsatisfactory definitions as were given by Euclid.

> **Discussion topic:** Why are Euclid's definitions of point and line unsatisfactory?

The modern axiomatic developments of Euclid's geometry usually take as undefined terms not only *point* and *line*, but also *lie on* (as in 'the point $P$ lies on the line'), *between* (as in 'on the line segment $AB$, $C$ lies between $A$ and $B$') and *congruent* (a means of comparing line segments without using a notion of length, and comparing angles without a measure corresponding to degrees or radians) ; see [6].

Having agreed on our logic and on our undefined terms, we can write down the axioms whose consequences we are to study. It then seems natural to require that these axioms satisfy as many of the following criteria as possible:

*Consistency:* it should not be possible to deduce a contradiction (such as '$0 = 1$') from the axioms.

*Adequacy:* we should be able to deduce from the axioms all the theorems that we want.

*Independence:* if we omit an axiom, then that axiom should not be derivable from the other axioms.

*Relevance:* the axioms should lead to interesting mathematics.

An example of inconsistency occurs in set theory. Towards the end of last century, the German logician Gottlob Frege produced the first axioms for the theory of sets. To Frege's dismay, just before his theory was published, Bertrand Russell pointed out to him that his axioms led to a contradiction. The problem arose with the following axiom, which we shall write in verbal, as distinct from Frege's symbolic, form:

> If $P$ is any property, then there is a set consisting of all the objects with the property $P$. (*)

Russell's contradiction (known generally as *Russell's paradox*[2]) deals with the case where $P$ is the property 'is a member of itself'. According to Frege's axiom (*), the set

$$S \equiv \{x : x \text{ is a set, and } x \text{ is not a member of } x\}$$

exists; call it $S$. A little thought should convince you that $S$ is a member of itself if and only if $S$ is not a member of itself. This is clearly absurd, and so if we accept (*) as an axiom of set theory, we have an inconsistent system of axioms.

> **Project:** What undefined terms and axioms are adopted in Zermelo–Fraenkel set theory, which for about a century has been the most widely accepted foundation of mathematics?

To illustrate the phenomenon of inadequacy, we consider axioms for the positive integers. The undefined terms here are *positive integer* and *successor*, where the successor of $1$ is $2$, that of $2$ is $3$, and so on. The axioms are:

**P1** $1$ is a positive integer.

**P2** Every positive integer has a unique successor, which is also a positive integer.

**P3** $1$ is not the successor of any positive integer.

**P4** Two positive integers with equal successors are themselves equal.

As they stand, these axioms are inadequate for the development of number theory, as we cannot prove from them anything which allows us to justify the method of induction. In order to arrive at an adequate system, we therefore introduce as an axiom the *Principle of Mathematical Induction*:

**P5** If a set $S$ of positive integers contains $1$, and contains the successor of $n$ whenever $n$ itself belongs to $S$, then $S$ is the set of all positive integers.

It turns out that these five axioms, first given by Giuseppe Peano in 1889, are adequate for the development of number theory.

Another example of inadequacy arises in geometry. Euclid's original axioms did not enable him to prove that every line contains at least one point. It appears that Euclid did not realise this gap in his theory, as he certainly assumed that lines did contain points when he proved his theorems.

Geometry also provides us with the most famous example of independence of axioms. One version of Euclid's axiom of parallels states that if $P$ is a point and $l$ is a line not passing through $P$, then there exists a unique line through $P$ parallel to $l$. For nearly two thousand years, mathematicians struggled to prove that this axiom was dependent on—that is, could be proved from—the other axioms of Euclid's geometry. Working separately in the early years of the nineteenth century, C.F.G Gauss, Janos Bolyai, and Nikolai Lobachevsky each developed a *non-Euclidean geometry*,

[2] For more on paradoxes see [13].

in which Euclid's axiom of parallels fails to hold but the other Euclidean axioms do hold. Of course, there then remained the possibility that the axioms of non-Euclidean geometry were inconsistent. Later that century, it was shown by E. Beltrami that, provided Euclid's original geometry is consistent, the non-Euclidean geometry of Gauss, Bolyai, and Lobachevsky is also consistent. It followed from this that Euclid's axiom of parallels could not be proved from the other axioms of Euclidean geometry.

Actually, independence of the axioms we study is not such an important requirement as some of the others. It may be unsatisfactory from an aesthetic point of view, but it is not incorrect in any way, to have some of our axioms provable as theorems from the others.

The notion of relevance of our axioms surely requires little further comment, except that we should realise that to describe a part of mathematics as interesting or significant is to make, not a provable assertion, but a value judgement. Do not expect all mathematicians to agree with you on what is, or is not, significant mathematics.

A more profound matter on which mathematicians are liable to disagree is the answer to the question 'What is mathematics?'.[3] One view was expressed, rather cynically, by Bertrand Russell:

> Mathematics is the subject in which we do not know what we are talking about, nor whether what we say is true.

This comment reflects a view shared by many, in which mathematics is seen as a formal game played with meaningless symbols according to certain conventions.[4] The best known exponent of this *formalist* philosophy of mathematics was David Hilbert (1862–1943), whose immensely fertile mind dominated mathematical thinking in the late nineteenth and early twentieth centuries.[5] It was Hilbert's belief that mathematics was the game of set theory, played with the rules of classical logic and with certain other rules laid down in the axioms for creating sets. Although these axioms referred to 'sets', Hilbert contended that, ultimately, there was no real meaning to the word 'set' used in mathematics: for him, that word could be replaced by 'beer mug', and the axioms would still be of interest and mathematical significance.

A particular aspect of Hilbert's philosophy was his interpretation of the phrase 'there exists'. For Hilbert, to prove that there exists a number with a certain property it was sufficient to assume that no such number exists, and then deduce a contradiction. However, such a procedure would not help us to calculate the number in question. This was one reason why Brouwer was so firmly opposed to Hilbert's formalist

[3] While it is not a deeply philosophical viewpoint, W.W. Sawyer's description of mathematics as 'the classification and study of all possible patterns' appeals to many practitioners of the subject. By 'pattern', Sawyer means 'almost any kind of regularity that can be recognized by the mind' [15, page 12].

[4] Actually, this was not quite Russell's view: he considered mathematics to be essentially the formal logic of propositions.

[5] See [14] for a very readable and interesting account of the life and times of Hilbert.

philosophy. Another reason for Brouwer's opposition was that Brouwer regarded mathematical objects as having a reality which Hilbert denied them. In particular, for Brouwer the positive integers were creations of the human intellect, and not meaningless symbols scratched on paper with a pen. Also, he believed that to justify the claim that there exists an object $x$ with the property $P$, we must provide a means of finding/calculating an object $x$ and showing algorithmically that it has that property. In 1930, Brouwer's successor in Amsterdam, Arend Heyting, captured the essence of Brouwer's methods by introducing *intuitionistic logic [10]*, intended as a replacement for classical logic; that logic is of considerable interest to logicians and computer scientists, but only a minority[6] of mathematicians today base their research on it.

For a long time, Hilbert hoped that it would be possible to prove the consistency of his formal game of mathematics by means acceptable to the followers of Brouwer's philosophy of *Intuitionism*.[7] He believed that if the consistency of axiomatic mathematics could be established by Brouwerian methods and reasoning wholly within the axiomatic system, then the use of his methods (such as that for proving existence by contradiction) would be justified. However, a result published by Kurt Gödel in 1931 showed that Hilbert's hopes were unfounded: very roughly, *Gödel's second incompleteness theorem* says that for any decent[8] set of axioms, the consistency of the resulting mathematics cannot be proved without resorting to reasoning outside the system itself.[9]

The controversy over the nature of mathematics was not ended by Gödel's theorem. Indeed, many mathematicians would argue that the justification of Hilbert's methods lies, not in any proof of the consistency of his axiom system, but in the enormous success of Hilbertian mathematics in the description of the physical universe.[10] As Hilbert himself said,

> [denying mathematicians the use of Hilbert's methods] would be the same, say, as proscribing the telescope to the astronomer, or to the boxer the use of his fists.

Another philosophy to which many mathematicians (at least, among those who think about the philosophy of their subject) subscribe is known as *Platonism.* Following the great Greek philosopher Plato and his mentor, Socrates, the Platonist[11] believes that mathematical objects have a real existence in their 'perfect forms', and that when we prove theorems about, say groups, we are really saying something about the perfect form of group. A proper discussion of Platonism requires more background

[6] A minority that has grown substantially in the last 60 years.

[7] For more on Brouwer's intuitionism, see [11]; pages 105–106 of that book present Heyting's original axioms for intuitionistic logic.

[8] We make no attempt to clarify what we mean here by 'reasonable', other than to say that it covers the current axioms for set theory, geometries, groups, ... .

[9] For a relatively accessible account of Gödel's theorems see [4, 12].

[10] Among the leading exponents of Hilbert's views were those mathematicians who wrote collectively under the pseudonym Nicolas Bourbaki [5, 8, 7].

[11] Gödel was one of the leading Platonists of his day.

than we can assume in this brief, rudimentary outline; suffice it to emphasise here that the platonist, like the intuitionist, regards mathematical objects as real entities, not simply meaningless symbols.

While it may be the case that a majority of practising mathematicians today would endorse either the Hilbert–Bourbaki philosophy of Formalism or else that of Platonism, there is an active minority group of researchers, following in the footsteps of Errett Bishop (1928–83) [1, 2], who adopt a philosophy akin to Brouwer's Intuitionism. Thus, in contrast to what many non-mathematicians believe, the nature and methodology of mathematics are not beyond dispute [9]. Ultimately, any philosophy of mathematics is a matter of opinion and value judgement, and is, or should be, open to reasonable discussion and reasoned criticism by the mathematical community.

**Discussion topics:**

1. In light of Brouwer's criticism of Hilbert's formalist mathematics, consider the following 'obvious' statement: If $a_1, a_2, a_3, \ldots$ is an infinite sequence of $0$'s and $1$'s,, then either $a_n = 0$ for all $n$, or there exists $n$ with $a_n = 1$. Discuss why this statement was not acceptable to Brouwer.
2. Discuss the distinction, if any, between pure and applied mathematics.
3. Why, if at all, does the nature of mathematics matter?

# References


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